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IGUSA'S LOCAL ZETA FUNCTIONS OF SEMIQUASIHOMOGENEOUS POLYNOMIALS

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ABSTRACT. In this paper, we prove the rationality of Igusa's local zeta functions of semiquasihomogeneous polynomials with coefficients in a non-archimedean local field K. The proof of this result is based on Igusa's stationary phase formula and some ideas on Néron π-desingularization.

1. INTRODUCTION

Let K be a non-archimedean local field, and let $\mathcal{O}_K$ be the ring of integers of K. Let π be a uniformizing parameter of K, and let the residue field of K be $\mathbb{F}_q$, the finite field with $q = p^r$ elements. Let v denote the valuation of K such that $v(\pi) = 1$. For $x \in K$, let $|x|_K = q^{-v(x)}$. Let $\mid dx \mid$ be the Haar measure on K^n, normalized so that the measure of $\mathcal{O}_K^n$ is equal to one. Let $f(x) \in K[x]$, $x = (x_1, .., x_n)$. The Igusa local zeta function associated to f is defined by

$$Z(f,s) = \int_{\mathcal{O}_K^n} |f(x)|_K^s \mid dx \mid, \quad s \in \mathbb{C},\ Re(s) > 0.$$

The local zeta function $Z(f,s)$ is a holomorphic function on the semiplane $Re(s) > 0$. In the case of K having characteristic zero, Igusa ([7], [8]) and Denef ([3]) proved that $Z(f,s)$ is a rational function of q^{-s}. At the present time, the techniques used by Igusa (resolution of singularities) and Denef (elimination of quantifiers in $\mathbb{Q}_p$) are not available in positive characteristic, so in this case the rationality of $Z(f,s)$ is still an open problem.

The local zeta function contains information about the number of solutions of the congruence $f(x) \equiv 0 \bmod \pi^j\mathcal{O}_K$ (see e.g. [4]). More precisely, if

$$N_j := \text{Card}\{x \in \left(\mathcal{O}_K/\pi^j\mathcal{O}_K\right)^n \mid f(x) \equiv 0 \mod \pi^j\mathcal{O}_K\},$$

and $P(t)$ is the Poincaré series $P(t) = \sum_{j=0}^{\infty} N_j (q^{-n}t)^j$, then

$$Z(f,s) = P(q^{-s}) - q^s(P(q^{-s}) - 1).$$

In this paper, we shall study the local zeta functions of semiquasihomogeneous polynomials with an absolutely algebraically isolated singularity at the origin of K^n.

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Let $f(x)$ be a polynomial with coefficients in K, and V_f the corresponding $K-$hypersurface. We call a point $P \in K^n$ an *absolutely algebraically isolated singularity* of $V_f(K)$, if the only solution of the system of equations

$$f(x) = \frac{\partial f}{\partial x_1}(x) = ... = \frac{\partial f}{\partial x_n}(x) = 0,$$

over an algebraic fixed closure of K, is the point P.

Let $\alpha_1, .., \alpha_n$ be n relatively prime and positive integers. A polynomial $f(x) \in K[x]$ is called a *quasihomogeneous polynomial* of weight d and exponents $\alpha_1, .., \alpha_n$, if it satisfies

$$f(t^{\alpha_1}x_1, .., t^{\alpha_n}x_n) = t^d f(x), \quad \text{for every } t \in K,$$

and the origin of K^n is an absolutely algebraically isolated singularity of the $K-$hypersurface V_f.

A polynomial $F(x)$ is called a *semiquasihomogeneous polynomial* if it has the form $f(x) + \sum b_i e_i(x) \in K[x]$, where $f(x)$ is a quasihomogeneous polynomial, and each monomial $e_i(x) = x_1^{m_1}...x_n^{m_n}$ satisfies $\sum_{j=1}^n \alpha_j m_j > d$, and the origin of K^n is an absolutely algebraically isolated singularity of the $K-$hypersurface V_F. We call the polynomial $f(x)$ the quasihomogeneous part of $F(x)$.

We put $|\alpha| = \sum_i \alpha_i$, for any $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{Z}^n$. We use the notation $Z(f, D, s)$ for the integral $\int_D |f(x)|_K^s \mid dx \mid$. In the case of $D = \mathcal{O}_K^n$, we use the simplified notation $Z(f, s)$.

The main result of this paper is the following:

Theorem 3.5. *Let $F(x) \in K[x]$ be a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, .., \alpha_n$. Then Igusa's local zeta function of $F(x)$ is a rational function of q^{-s}. More precisely,*

$$Z(F, s) = \frac{L(q^{-s})}{(1 - q^{-1}q^{-s})(1 - q^{-|\alpha|}q^{-ds})}, \tag{1.1}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$. Furthermore, the polynomial $L(q^{-s})$ can be computed effectively.

If in addition the polynomial F is nondegenerate for its Newton diagram and if K has characteristic zero, then a very different way of calculating $Z(F, s)$ is given in [5]. The proof of theorem 3.5 gives an effective method to compute the local zeta functions of semiquasihomogeneous polynomials.

We say that a singular point $\bar{P} \in V_{\bar{f}}(\mathbb{F}_q)$, where $\overline{f}$ is the reduction modulo π of f, is a non-liftable singularity of the hypersurface V_f, if for every singular point $Q \in V_f(K)$, $Q \in \mathcal{O}_K^n$, the reduction modulo π of Q is different from $\overline{P}$. The proof of theorem 3.5 shows that the numerator of the zeta function $Z(F, s)$ depends on the non-liftable singularites of the closed fiber of the hypersurface V_F, and the denominator depends on the singularity of the generic fiber of V_F. More precisely, the denominator depends on Newton's diagram of $F(x)$. In the proof of theorem 3.5, we use Igusa's formula of stationary phase for π-adic integrals ([10]) and some ideas on Néron π-desingularization ([13], sect. 17, 18).

As a consequence of theorem 3.5, we obtain the following three corollaries.

Corollary 3.6. *Let K be a global field and $F(x) \in K[x]$ a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, .., \alpha_n$. Then for every non-archimedean valuation v of K, Igusa's local zeta function of*

$F(x)$ on the completion K_v of K is a rational function of the form (1.1). If K is a number field and $F(x)$ is non-degenerate for its Newton's diagram, then the real parts of the poles of the zeta function $Z(F,s)$ are roots of the Bernstein polynomial of $F(x)$.

For the definition of the Bernstein polynomial and its computation in the non-degenerate case, see reference [2]. The last part of corollary 3.6 is a special case of a more general result due to Loeser (cf. [11], thm. 5.5.1).

Corollary 3.7. *Let K be a global field and $\mathcal{O}_K$ its ring of integers. Let $F(x) \in \mathcal{O}_K[x]$ be a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, ..., \alpha_n$. Then for every non-archimedean valuation v of K, the number of solutions $N_j(F,v)$ of the congruence*

$$F(x) \equiv 0 \ \ mod \ \ \pi^j \mathcal{O}_{K_v},$$

where $\mathcal{O}_{K_v}$ is the ring of integers of the completion K_v, satisfies

$$l\acute{\imath}m\ sup_{j\to\infty} N_j(F,v)^{1/j} \leq \begin{cases} q^{n-|\alpha|/d} & if \quad |\ \alpha\ | /d \leq 1, \\ q^{n-1} & if \quad |\ \alpha\ | /d > 1. \end{cases}$$

The zeta functions of plane curves, with only an absolutely analytically irreducible singularity at the origin, have been extensively studied when the characteristic of K is zero, by Igusa ([9]), Meuser ([12]), among others. Let $f(x,y) \in K[x,y]$ be an absolutely analytically irreducible polynomial. Thus there exist $(\alpha_1, \alpha_2) \in \mathbb{N}^2$, relatively prime integers, and an integer d, such that $f(x,y) = f_d(x,y) + g(x,y)$ and the monomials $x^n y^m$ of $f_d(x,y)$ and $g(x,y)$ satisfy $\alpha_1 n + \alpha_2 m = d$ and $\alpha_1 n + \alpha_2 m > d$, respectively. The origin of K^2 is an absolutely algebraically isolated singularity of the plane curve V_f. If it is also valid for V_{f_d}, then f is a semiquasihomogeneous polynomial in the sense of the definition given for us. As a consequence of theorem 3.5, we obtain a precise description of the local zeta function associated with this type of polynomials.

Corollary 3.8. *Let $f(x,y) \in K[x,y]$ be an absolutely analytically irreducible polynomial, such that the origin of K^2 is an absolutely algebraically isolated singularity of the plane curve V_{f_d}. Then the Igusa local zeta function $Z(f,s)$ is a rational function of q^{-s} of the form (1.1).*

2. Preliminaires

In [10] Igusa introduced the stationary phase formula for $\pi-$adic integrals and suggested that a closer examination of this formula might lead to a proof of the rationality of $Z(f,s)$ in any characteristic. That suggestion has been our main motivation for this paper. In this section we review Igusa's stationary phase formula and some ideas on Néron $\pi-$desingularization.

We denote by $\overline{x}$ the image of an element of $\mathcal{O}_K$ under the canonical homomorphism $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$, i.e, the reduction modulo π. Given $f(x) \in \mathcal{O}_K[x]$ such that not all its coefficients are in $\pi\mathcal{O}_K$, we denote by $\overline{f(x)}$ the polynomial obtained by reducing modulo π the coefficients of $f(x)$.

For any commutative ring A and $f(x) \in A[x]$, we denote by $V_f(A)$ the set of $A-$valued points of the hypersurface V_f defined by f, and by $Sing_f(A)$ the set of

$A-$valued singular points of V_f, i.e.,

$$Sing_f(A) = \{x \in A^n \mid f(x) = \frac{\partial f}{\partial x_1}(x) = ... = \frac{\partial f}{\partial x_n}(x) = 0\}.$$

We fix a lifting R of $\mathbb{F}_q$ in $\mathcal{O}_K$. Thus, the set R is mapped bijectively onto $\mathbb{F}_q$ by the canonical homomorphism $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K$.

Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial in n variables, $P_1 = (y_1, ..., y_n) \in \mathcal{O}_K^n$, and $m_{P_1} = (m_1, ..., m_n) \in \mathbb{N}^n$. We call a K^n-isomorphism $\psi_{m_{P_1}}(x)$ a *dilatation*, if it satisfies $\psi_{m_{P_i}}(x) = (z_1, .., z_n)$, $z_i = y_i + \pi^{m_i} x_i$, for each $i = 1, 2, .., n$. We define the *dilatation* of $f(x)$ at P_1 induced by $\psi_{m_{P_1}}(x)$, as

$$f_{P_1}(x) := \pi^{-e_{P_1}} f(\psi_{m_{P_1}}(x)), \tag{2.1}$$

where e_{P_1} is the minimum order of π in the coefficients of $f(\psi_{m_{P_1}}(x))$. We call the $K-$hypersurface $V_{f_{P_1}}$ the dilatation of V_f at P_1 induced by $\psi_{m_{P_1}}(x)$, the number e_{P_1} *the arithmetic multiplicity of* $f(x)$ *at* P_1 by $\psi_{m_{P_1}}(x)$ and the set $S(f_{P_1})$, the lifting of $Sing_{\overline{f_{P_1}}}(\mathbb{F}_q)$, *the first generation of descendants of* P_1.

Given a sequence of dilatations $(\psi_{m_{P_k}}(x))_k$, we inductively define $e_{P_1,...,P_k}$, $f_{P_1,...,P_k(x)}$, and $S(f_{P_1,...,P_k})$ as follows:

$$f_{P_1,...,P_k}(x) = \begin{cases} f(x) & \text{if } k = 0, \\ \pi^{-e_{P_1,...,P_k}} f_{P_1,...,P_{k-1}}(\psi_{m_{P_k}}(x)) & \text{if } k \geqq 1, \end{cases} \tag{2.2}$$

where $P_k \in S(f_{P_1,...,P_{k-1}})$, and $e_{P_1,...,P_k}$ is the minimum order of π in the coefficients of $f_{P_1,...,P_{k-1}}(\psi_{m_{P_k}}(x))$. The union of the sets $S(f_{P_1,...,P_k})$ is called *the* $k-$*generation of descendants of* P_1.

In this paper we shall use several types of dilatations, i.e., dilatations with different m's, but the specific value of m will be clear from the context. The dilatations were introduced by A. Néron (cf. [13], sect. 18). These transformations play an important role in the process of desingularization of the closed fiber of a scheme over a discrete valuation ring whose generic fiber is non-singular. A modern exposition of the Néron $\pi-$desingularization can be found in [1], sect. 4.

Now, we review Igusa's stationary phase formula, from the point of view of the dilatations. For that, we fix the m_{P_k}'s equal to $(1, .., 1) \in \mathbb{N}^n$ in (2.2).

Let $\overline{D}$ be a subset of $\mathbb{F}_q^n$ and D its preimage under the canonical homomorphism $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$. Let $S(f, D)$ denote the subset of R^n (the set of representatives of $\mathbb{F}_q^n$ in $\mathcal{O}_K^n$) mapped bijectively to the set $Sing_{\bar{f}}(\mathbb{F}_q) \cap \overline{D}$. We use the simplified notation $S(f)$, in the case of $D = \mathcal{O}_K^n$. Also we define

$$\nu(\bar{f}, D) := q^{-n}\text{Card}\{\overline{P} \in \overline{D} | \ \overline{P} \notin V_{\bar{f}}(\mathbb{F}_q)\},$$
$$\sigma(\bar{f}, D) := q^{-n}\text{Card}\{\overline{P} \in \overline{D} | \ \overline{P} \ \text{ is a non-singular point of } \ V_{\bar{f}}(\mathbb{F}_q)\}.$$

In order to simplify the notation, we shall use $\nu(\bar{f})$ and $\sigma(\bar{f})$ instead of $\nu(\bar{f}, D)$ and $\sigma(\bar{f}, D)$, respectively, if $D = \mathcal{O}_K^n$.

With all this, we are able to establish Igusa's stationary phase formula for $\pi-$adic integrals:

Igusa's Stationary Phase Formula ([10, p. 177]).

$$\int_D |f(x)|_K^s \mid dx \mid = \nu(\bar{f}, D) + \sigma(\bar{f}, D)\frac{(1-q^{-1})q^{-s}}{(1-q^{-1}q^{-s})} + \sum_{P\in S(f,D)} q^{-n-e_P s}\int_{\mathcal{O}_K^n} |f_P(x)|_K^s \mid dx \mid, \tag{2.3}$$

where $Re(s) > 0$. Formula (2.3) is obtained in the following form. Suppose that $\overline{D} = \{\overline{P_1}, .., \overline{P_N}\}$ and let P_i be the lifting of $\overline{P_i}$. Then the set D is the disjoint union $\bigcup_P D_P$, where $\overline{P} = (\overline{y_1}, .., \overline{y_n}) \in \overline{D}$ and D_P is defined as

$$D_P = \{(x_1, ..., x_n) \in D \mid x_i = y_i + \pi z_i,\ z_i \in \mathcal{O}_K,\ \ i = 1, 2, .., n\}.$$

Thus

$$\int_D |f(x)|_K^s \mid dx \mid = \sum_{\bar{P}} \int_{D_P} |f(x)|_K^s \mid dx \mid = \sum_{\bar{P}} q^{-n-e_P s}\int_{\mathcal{O}_K^n} |f_P(x)|_K^s \mid dx \mid .$$

The integrals corresponding to the P's for which $\overline{P} \notin V_{\bar{f}}(\mathbb{F}_q)$ are easily computable. The integrals corresponding to the P's for which $\overline{P}$ is a non-singular point of $V_{\bar{f}}(\mathbb{F}_q)$ are computed using the implicit function theorem (cf. [10], p. 177).

We define I_k inductively as follows:

$$I_1 := \{P_1 \mid\ P_1 \in S(f, D)\},$$

and

$$I_k := \{(P_1, .., P_k) \mid\ (P_1, .., P_{k-1}) \in I_{k-1} \text{ and } P_k \in S(f_{P_1,..,P_{k-1}})\}, \quad k \geq 2.$$

With the above notation and by iterating the stationary phase formula, we obtain the following expansion for $Z(f, s)$ (cf. [10], p. 178):

$$\begin{aligned} Z(f,s) = {} & \nu(\bar{f}, D) + \sigma(\bar{f}, D)\frac{(1-q^{-1})q^{-s}}{(1-q^{-1}q^{-s})} \\ & + \sum_{k\geq 1} q^{-kn}\left(\sum_{(P_1,\dots P_k)\in I_k} \nu(\overline{f}_{P_1,\dots P_k})q^{-E(P_1,\dots P_k)s}\right) \\ & + \frac{(1-q^{-1})q^{-s}}{(1-q^{-1}q^{-s})}\sum_{k\geq 1} q^{-kn}\left(\sum_{(P_1,\dots P_k)\in I_k} \sigma(\overline{f}_{P_1,\dots P_k})q^{-E(P_1,\dots P_k)s}\right), \end{aligned} \tag{2.4}$$

where $E(P_1, .., P_k) := e_{P_1} + ... + e_{P_1,...P_k}$. Expansion (2.4) converges absolutely on the semiplane $Re(s) > 0$.

Now, we summarize some ideas on Néron $\pi-$desingularization (see [13], sect. 17, 18) to be used in the next sections. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial, and $P \in V_f(\mathcal{O}_K)$. Néron introduced the following measure of singularity at P:

$$l(f, P) := Inf_{i=1,..,n}\left(v(\frac{\partial f}{\partial x_i}(P))\right).$$

The Jacobian criterion implies that P is a smooth point of $V_f(K)$ if and only if $l(f, P)$ is finite. $\overline{P}$ is a smooth point of $V_{\bar{f}}(\mathbb{F}_q)$ if and only if $l(f, P) = 0$. In this paper, we introduce the following measure of singularity at an integer point P, satisfying $P \notin Sing_f(\mathcal{O}_K)$.

Definition 2.1. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial and $P \in \mathcal{O}_K^n$ a point, such that $P \notin Sing_f(\mathcal{O}_K)$. We define

$$L(f,P) := Inf\left(v(f(P)), v(\frac{\partial f}{\partial x_1}(P)), ..., v(\frac{\partial f}{\partial x_n}(P))\right).$$

Let us observe that $L(f,P)=0$ if and only if

$$\overline{f(x)} = \alpha_0 + \sum_{j=1}^{n} \alpha_j (x_j - \overline{a_j}) + (\text{degree } \geq 2),$$

where $P=(a_1,..,a_n)$, $\alpha_j \in \mathbb{F}_q^*$ for some $j=0,1,2,..,n$. We also observe that $l(f,p) = L(f,p)$ if $P \in V_f(\mathcal{O}_K)$. The integer $L(f,P)$ has properties similar to those of $l(f,P)$. This integer appears naturally associated to Igusa's stationary phase, as we shall see later on.

We denote by A_r, $r=(r_1,..,r_n) \in (\mathbb{N}\setminus\{0\})^n$, the set

$$A_r := \{x \in \mathcal{O}_K^n \ \mid \ v(x_i) \geq r_i, \ i=1,..,n\}.$$

From a geometrical point of view, A_r is a polydisc in $\mathcal{O}_K^n$ centered at the origin. The complement of A_r in $\mathcal{O}_K^n$ is denoted by A_r^c.

The following proposition is a simple reformulation of proposition 17 in sect. 17 of [13]. However, for our convenience, we prove it below.

Proposition 2.2 (Néron, [13], sect. 17, prop. 17)**.** *Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial, $P \in \mathcal{O}_K^n$ an absolutely algebraically isolated singularity of the hypersurface V_f, and let $D \subseteq \mathcal{O}_K^n$ be a subset such that $D \cap (P + A_r) = \emptyset$, for some $r \in (\mathbb{N} \setminus \{0\})^n$. Then*

$$L(f,Q) \leq C(f,D), \ \textit{for every} \ \ Q \in D,$$

where the constant $C(f,D)$ depends only on f and D.

Proof. Without loss of generality, we may suppose that the point P is the origin of K^n. The hypothesis that the origin of K^n is an absolutely algebraically isolated singularity and the Hilbert Nullstellensatz imply that

$$\pi^{m_i} x_i^{t_i} = A_{i,0}(x) f(x) + \sum_{j=1}^{n} A_{i,j}(x) \frac{\partial f}{\partial x_j}(x), \tag{2.5}$$

for some m_i, $t_i \in \mathbb{N}$, with $t_i \neq 0$, and some polynomials $A(x)_{i,j} \in \mathcal{O}_K[x]$, for each $i=1,2,..,n$. Now, let $Q=(q_1,..,q_n)$ be a point of D. Since $D \cap A_r = \emptyset$, there exists a coordinate j_0 such that $v(q_{j_0}) < r_{j_0}$. From (2.5), with $x=Q$ and $i=j_0$, we obtain

$$m_{j_0} + t_{j_0} r_{j_0} \geq m_{j_0} + t_{j_0} v(q_{j_0}) \geq L(f,Q).$$

Thus, it is sufficient to take $C(f,D) \geq Max_i\{m_i + t_i r_i\}$. □

The following result is a generalization of proposition 18 (cf. [13], sect. 18) of A. Néron.

Proposition 2.3 (Néron, [13], sect. 18, prop. 18)**.** *Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial and $P=(b_1,..,b_n) \in \mathcal{O}_K^n$ a point such that $P \notin Sing_f(\mathcal{O}_K)$. Then there exists a minimal non-negative integer $\mu(f,P)$ such that the polynomial*

$$f_P(x) = \pi^{-e_{\mu,P}} f(P + \pi^{\mu} x),$$

where $e_{\mu,P}$ is the minimum order of π in the coefficients of $f(P+\pi^{\mu}x)$, satisfies

$$\overline{f_P(x)} = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \in \mathbb{F}_q^*, \text{ for some } i = 0, 1, 2, .., n.$$

Furthermore, $\mu(f,P) \leq L(f,P)+1$.

Proof. By induction on $L(f,P)$.

Case $L(f,P)=0$. In this case, we take $\mu(f,P)=1 \leq L(f,P)+1$.

Case $L(f,P)>0$. In this case, we have

$$f(x) = \alpha_0 + \sum_{i=1}^{n} \alpha_i (x_i - b_i) + (\text{degree} \geq 2),$$

where $\alpha_i \equiv 0 \bmod \pi$, $i = 0, 1, .., n$. Thus

$$f(P+\pi x) = \pi \left(\alpha_0' + \sum_{i=1}^{n} \alpha_i x_i + \pi(\text{degree} \geq 2) \right).$$

We consider two cases according to whether $\alpha_0' \not\equiv 0 \bmod \pi$ or not.

Case 1 ($\alpha_0' \not\equiv 0 \bmod \pi$). In this case, we have

$$f(P+\pi x) = \pi f_P(x),$$

where

$$f_P(x) = \alpha_0' + \sum_{i=1}^{n} \alpha_i x_i + \pi(\text{degree} \geq 2).$$

Therefore, $\overline{f_P(x)} = \overline{\alpha_0'} \in \mathbb{F}_q^*$, and $\mu(f,P) = 1 \leq L(f,P)$.

Case 2 ($\alpha_0' \equiv 0 \bmod \pi$). In this case, we have

$$f(P+\pi x) = \pi^2 \left(\alpha_0'' + \sum_{i=1}^{n} \alpha_i' x_i + (\text{degree} \geq 2) \right) = \pi^{e_{\mu,P}} f_P(x),$$

where $e_{\mu,P} \geq 2$. Thus

$$f(P) = \pi^{e_{\mu,P}} f_P(0),$$

and

$$\frac{\partial f}{\partial x_i}(P) = \pi^{e_{\mu,P}-1} \frac{\partial f_P}{\partial x_i}(0), \quad i = 1, .., n.$$

Hence $L(f_P, 0) \leq L(f,P) - 1$. The result follows from the induction hypothesis by observing that $\mu(f,P) = \mu(f_P,0)+1 \leq L(f_P,0)+1+1 \leq L(f,P)+1$. □

As a consequence of the two above results, we obtain the following lemma.

Lemma 2.4. *Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial such that the origin of K^n is an absolutely algebraically isolated singularity. Let A_r be a polydisc with $r \in (\mathbb{N} \smallsetminus \{0\})^n$. Then there exists $\gamma = \gamma(f,r) \in \mathbb{N} \smallsetminus \{0\}$, such that the polynomial*

$$f_Q(x) = \pi^{-e_{Q,\gamma}} f(Q + \pi^{\gamma} x),$$

satisfies the condition that $\overline{f_Q(x)}$ is a non-zero constant or a linear polynomial with or without constant term, for all $Q \in A_r^c$.

3. Rationality of Igusa's local zeta functions of semiquasihomogeneous polynomials

In this section, we prove the rationality of the Igusa local zeta function of semiquasihomogeneous polynomials.

Lemma 3.1. *Let $D \subseteq \mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\overline{D} \subseteq \mathbb{F}_q^n$. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial such that the origin of K^n is an absolutely algebraically isolated singularity of $V_f(K)$. If $D \cap A_r = \emptyset$, for some $r \in (\mathbb{N} \setminus \{0\})^n$, then the integral $Z(f, D, s) = \int_D |f|_K^s \mid dx \mid$ is a rational function of q^{-s}. More precisely,*

$$Z(f, D, s) = \frac{L(q^{-s}, D)}{1 - q^{-1}q^{-s}}. \tag{3.1}$$

Furthermore, the polynomial $L(q^{-s}, D)$ can be effectively computed.

Proof. By applying the stationary phase formula $m + 1 = \gamma(f, r)$ times, we obtain

$$Z(f, D, s) = \nu(\overline{f}, D) + \sigma(\overline{f}, D)\frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1}q^{-s})}$$

$$+ \sum_{k=1}^{m} q^{-kn} \left(\sum_{(P_1, P_2, \ldots P_k) \in I_k} \nu(\overline{f}_{P_1, \ldots P_k}) q^{-E(P_1, \ldots P_k)s} \right)$$

$$+ \frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1}q^{-s})} \sum_{k=1}^{m} q^{-kn} \left(\sum_{(P_1, P_2, \ldots P_k) \in I_k} \sigma(\overline{f}_{P_1, \ldots P_k}) q^{-E(P_1, \ldots P_k)s} \right)$$

$$+ \sum_{(P_1, .., P_{m+1}) \in I_{m+1}} q^{-(m+1)n - E(P_1, \ldots P_{m+1})s} \int_{\mathcal{O}_K^n} |f_{P_1, .., P_{m+1}}(x)|_K^s \mid dx \mid . \tag{3.2}$$

On the other hand, we have

$$f(P_1 + P_2\pi + ... + P_m\pi^{m+1} + \pi^{m+2}x) = \pi^{E(P_1, \ldots, P_m)} f_{P_1, \ldots., P_m}(x), \tag{3.3}$$

and $P_1 + P_2\pi + ... + P_m\pi^{m+1} \in A_r^c$. Because $m + 2 \geq \gamma(f, r)$, it follows from lemma 2.4 and (3.3) that the $\mathbb{F}_q-$hypersurface defined by $\overline{f_{P_1, \ldots., P_m}(x)}$ is smooth or empty. Then $I_{m+1} = \emptyset$, and the integral $Z(f, D, s)$ is a rational function of form (3.1). ☐

Lemma 3.2. *Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial such that the origin of K^n is an absolutely algebraically isolated singularity of $V_f(K)$. Then the integral $Z(f, A_r^c, s) = \int_{A_r^c} |f|^s \mid dx \mid$ is a rational function of q^{-s}. More precisely,*

$$Z(f, A_r^c, s) = \frac{L(q^{-s})}{1 - q^{-1}q^{-s}}. \tag{3.4}$$

Furthermore, the polynomial $L(q^{-s})$ can be computed effectively.

Proof. We introduce a family $\mathcal{L}$ of sets defined as follows. For each subset B of $\{1, ..., n\}$ and each $a = (a_1, ..., a_n) \in \mathbb{N}^n$ satisfying $0 \leq a_i < r_i$ if $i \in B$, we define

$$D(B, a) := \begin{cases} \{x \in A_r^c \mid \ v(x_i) = a_i \text{ if } i \in B\}, & \text{if } B \neq \emptyset, \\ \emptyset, & \text{if } B = \emptyset. \end{cases} \tag{3.5}$$

The family $\mathcal{L}$ is closed under intersections, and its union is A_r^c. We denote by J the set of indices $\{(B,a)\}$ and by $\mathcal{P}(J)$ the family of subsets of J with i elements.

Hence

$$Z(f, A_r^c, s) = \sum_{i=1}^{\mathrm{Card}\{J\}} (-1)^{i-1} \sum_{T \in \mathcal{P}(J)_i} \int_{D(T)} |f|^s \mid dx \mid, \tag{3.6}$$

where $D(T) := \bigcap_{(B,a)\in T} D(B,a)$. From (3.6) and the fact that the family $\mathcal{L}$ is closed under intersections, it follows that in order to prove the theorem, it is sufficient to prove that any integral of type

$$\int_{D(B,a)} |f|^s \mid dx \mid, \quad B \neq \emptyset, \tag{3.7}$$

is a rational function of the form $L(q^{-s}, D(B,a))/(1-q^{-1}q^{-s})$, where the numerator polynomial is effectively computable. For that, we make the following change of variables in (3.7):

$$x = \psi_{(B,a)}(y), \text{ where } x_i = \begin{cases} \pi^{a_i} y_i & \text{if } i \in B, \\ y_i & \text{if } i \notin B. \end{cases} \tag{3.8}$$

We obtain

$$\int_{D(B,a)} |f|^s \mid dx \mid = q^{-e_{(B,a)}s - d_{(B,a)}} \int_{D'(B,a)} |f_B|^s \mid dy \mid, \tag{3.9}$$

where $d_{(B,a)} = \sum_{i\in B} a_i$, $f_B(y)$ is the dilatation of f induced by (3.8) and

$$D'(B,a) = \prod_{i=1}^{n} R_i,$$

where $R_i = \mathcal{O}_K$ if $i \notin B$ and $R_i = \mathcal{O}_K^*$ if $i \in B$.

On the other hand, $\phi(y)$ defines a $K-$isomorphism of $K^n \longrightarrow K^n$; thus the $K-$singular locus of V_f is mapped bijectively on the $K-$singular locus of V_{f_B}. Therefore, the polynomial f_B and the set $D'(B,a)$, $B \neq \emptyset$, satisfy the conditions of lemma 3.1. Thus the integral $\int_{D'(B,a)} |f_B|^s \mid dx \mid$ is a rational function of q^{-s}, and its numerator can be computed effectively. □

Proposition 3.3. *Let $F(x) = f(x) + \pi^m g(x) \in \mathcal{O}_K[x]$ be a polynomial such that the origin of K^n is an absolutely algebraically isolated singularity of $V_F(K)$. Let $D \subseteq \mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\overline{D} \subseteq \mathbb{F}_q^n$, and A_r a polydisc such that $D \cap A_r = \emptyset$, $r \in (\mathbb{N} \smallsetminus \{0\})^n$. There exists $\alpha(f,D)$, effectively computable, such that if $m \geq \alpha(f,D)$ then*

$$Z(F,D,s) = Z(f,D,s).$$

Proof. By virtue of lemma 2.4, there exists a $\gamma_0 = \gamma(F,r)$ such that the reduction modulo π of the polynomial

$$F_{P_1,\ldots,P_n}(x) = \pi^{-E_F(P_1,\ldots,P_n)} F(P_1 + P_2\pi + \ldots. + P_n\pi^n + \pi^{n+1}x) \tag{3.10}$$

is a non-zero constant or a linear polynomial for every $n \geq \gamma_0$ and any $P_1 + P_2\pi + \ldots. + P_n\pi^n \in A_r^c$. This implies implies that $I_{\gamma_0+1} = \emptyset$. So the integral $Z(F,D,s)$ admits a finite expansion of type (3.2).

We set

$$g_{P_1,\ldots,P_n}(x) := \pi^{-E_g(P_1,\ldots,P_n)} g(P_1 + P_2\pi + \ldots. + P_n\pi^n + \pi^{n+1}x),$$

where $E_g(P_1,..,P_n)$ is the minimum order of π in the coefficients of

$$g(P_1 + P_2\pi + ... + P_n\pi^n + \pi^{n+1}x).$$

With this notation, we have that

$$F(P_1 + P_2\pi + ... + P_n\pi^n + \pi^{n+1}x) = \pi^{E_f(P_1,..,P_n)} f_{P_1,...,P_n}(x)$$

$$+\pi^{E_g(P_1,..,P_n)+m} g_{P_1,...,P_n}(x). \tag{3.11}$$

Now, we choose

$$\alpha(f, D) := Max_{(P_1,..,P_{\gamma_0})\in I_{\gamma_0}} \{1 + E_f(P_1,..,P_{\gamma_0})\}. \tag{3.12}$$

Thus, if $m \geq \alpha(f, D)$, then

$$\overline{F_{P_1,P_2,..,P_k}(x)} = \overline{f_{P_1,P_2,..,P_k}(x)},$$

$$E_F(P_1, P_2,.., P_k) = E_f(P_1, P_2,.., P_k), \quad 1 \leq k \leq \gamma_0. \tag{3.13}$$

From (3.13) it follows that

1) $\nu(\overline{f}, D) = \nu(\overline{F}, D)$ and $I_1(f) = I_1(F)$, because $\overline{f}(x) = \overline{F}(x)$.

2) For every $1 \leq k \leq \gamma_0$

$$\nu(\overline{f_{P_1,..,P_k}}) = \nu(\overline{F_{P_1,..,P_k}}), \text{ for every } (P_1,.., P_k) \in I_k(f) = I_k(F).$$

3) For every $1 \leq k \leq \gamma_0$

$$I_{k+1}(f) = I_{k+1}(F),$$

because $I_k(f) = I_k(F)$ and $\overline{f_{P_1,..,P_k}(x)} = \overline{F_{P_1,..,P_k}}(x)$ for every $(P_1,.., P_k) \in I_k(f) = I_k(F)$.

Therefore $I_{\gamma_0+1}(f) = I_{\gamma_0+1}(F) = \emptyset$, and $Z(f, s, D)$ admits a finite expansion of type (3.2). Now it is easy to see that $Z(F, D, s) = Z(f, D, s)$. □

Lemma 3.4. *Let $F(x) = f(x) + \pi^m g(x) \in \mathcal{O}_K[x]$ be a polynomial such that the origin of K^n is an absolutely algebraically isolated singularity of $V_F(K)$. Let A_r be a polydisc, with $r \in (\mathbb{N} \setminus \{0\})^n$. There exists $\alpha(f, r)$, effectively computable, such that if $m \geq \alpha(f, r)$ then*

$$Z(F, A_r^c, s) = Z(f, A_r^c, s).$$

Proof. By (3.6), it is sufficient to prove the lemma for integrals of the type (3.7). We choose $\alpha(f, r)$ satisfying

$$\alpha(f, r) \geqq Max_{(B,a)}\{e_{f,(B,a)}\}.$$

With the above condition, it is sufficient to prove that

$$\int_{D'(B,a)} |F_B|^s \mid dx \mid = \int_{D'(B,a)} |f_B|^s \mid dx \mid, \tag{3.14}$$

with $D'(B, a)$ as in (3.9) and $F_B(x) = f_B(x) + \pi^{m-e_{f,(B,a)}} g(\psi_{(B,a)}(x))$.

The result follows from (3.14) and proposition 3.3. Finally, we observe that $\alpha(f, r)$ is given by

$$\alpha(f, r) = Max_{(B,a)}\{e_{f,(B,a)} + \alpha(f_B, D'(B, a))\}.$$

□

Theorem 3.5. *Let $F(x) \in K[x]$ be a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, .., \alpha_n$. Then Igusa's local zeta function of $F(x)$ is a rational function of q^{-s}. More precisely,*

$$Z(F,s) = \frac{L(q^{-s})}{(1-q^{-1}q^{-s})(1-q^{-|\alpha|}q^{-ds})} \tag{3.15}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$. Furthermore, the polynomial $L(q^{-s})$ can be computed effectively.

Proof. We may suppose without loss of generality that $F(x) \in \mathcal{O}_K[x]$. By decomposing $\mathcal{O}_K^n$ as the disjoint union of A_α and A_α^c and using the fact that $F(x)$ is a semiquasihomogeneous polynomial, we obtain

$$\begin{aligned} Z(F,s) &= \int_{A_\alpha} |F|_K^s \mid dx \mid + \int_{A_\alpha^c} |F|_K^s \mid dx \mid \\ &= q^{-|\alpha|-ds} \int_{\mathcal{O}_K^n} |F_1|_K^s \mid dx \mid + \int_{A_\alpha^c} |F|_K^s \mid dx \mid, \end{aligned} \tag{3.16}$$

where $F_1(x) = f(x) + \pi^{m_1} H_1(x)$, with $m_1 \geq 1$. Now, iterating formula (3.16) $m+1$ times, we obtain

(3.17)

$$Z(F,s) = Z(F, A_\alpha^c, s) + \sum_{k=1}^{m} q^{k(-|\alpha|-ds)} Z(F_k, A_\alpha^c, s) + q^{(m+1)(-|\alpha|-ds)} Z(F_{m+1}, s),$$

where $F_k(x) = f(x) + \pi^{m_k} H_k(x)$, with $m_k \longrightarrow \infty$. By lemma 3.4, there exists $\gamma_0 = \gamma_0(f, \alpha)$, effectively computable, such that

$$Z(F_k, A_\alpha^c, s) = Z(f, A_\alpha^c, s), \text{ for every } \ m_k \geq \gamma_0.$$

Thus from (3.17), we have

$$\begin{aligned} Z(F,s) = Z(F, A_\alpha^c, s) &+ \sum_{k=1}^{\gamma_0 - 1} q^{k(-|\alpha|-ds)} Z(F_k, A_\alpha^c, s) \\ &+ Z(f, A_\alpha^c, s) q^{(\gamma_0+1)(-|\alpha|-ds)} \frac{1}{1-q^{-|\alpha|-ds}}. \end{aligned} \tag{3.18}$$

By lemma 3.2 the integrals $Z(f, A_\alpha^c, s)$ and $Z(F_k, A_\alpha^c, s)$ are rational functions of q^{-s} of the form $\frac{L(q^{-s})}{1-q^{-1}q^{-s}}$, where the polynomial numerator can be effectively computed. □

We observe that if f is a quasihomogeneous polynomial, its local zeta function is given by $Z(f,s) = \frac{Z(f, A_\alpha^c, s)}{1-q^{-|\alpha|-ds}}$. The integral $Z(f, A_\alpha^c, s)$ can be computed using lemma 3.2.

As a consequence of theorem 3.5, we obtain the following three corollaries. In the first corollary we shall use freely the notions of Newton polyhedron, non-degeneracy with respect to a Newton polyhedron, and Bernstein polynomial; the corresponding definitions can be found in references [1], [2].

Corollary 3.6. *Let K be a global field and let $F(x) \in K[x]$ be a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, .., \alpha_n$. Then for every non-archimedean valuation v of K, Igusa's local zeta function of $F(x)$ on the completion K_v of K is a rational function of the form*

(3.15). If K is a number field and $F(x)$ is non-degenerate with respect to its Newton polyhedron, then the real parts of the poles of the zeta function $Z(F,s)$ are roots of the Bernstein polynomial of $F(x)$.

Proof. Since $F(x)$ has an absolutely algebraically isolated singularity at the origin of K^n, the Hilbert Nullstellensatz implies that for all non-archimedean valuations v of K, the origin of K_v^n is an absolutely algebraically isolated singularity of $V_f(K_v)$. Thus by the proof of theorem 3.5, the denominator of the the local zeta function $Z(f,s)$ on K_v is equal to $(1-q^{-1}q^{-s})(1-q^{-|\alpha|}q^{-ds})$. Thus the real parts of the poles of $Z(F,s)$ are among the values $-1, -\mid \alpha \mid /d$. If K is a number field, and $F(x)$ is non-degenerate with respect to its Newton polyhedron, theorem C.2.2.3 of [2] implies that -1 and $-\mid \alpha \mid /d$ are roots of the Bernstein polynomial of $F(x)$. □

The following corollary gives a bound for the number of solutions of a conguence attached to a semiquasihomogeneous polynomial with coefficients in a ring of integers of a global field. This corollary follows directly from the relation existing between the Igusa local zeta function and the Poincaré series $P(t)$ (see the introduction) and corollary 3.6.

Corollary 3.7. *Let K be a global field and $\mathcal{O}_K$ its ring of integers. Let $F(x) \in \mathcal{O}_K[x]$ be a semiquasihomogeneous polynomial whose quasihomogeneous part $f(x)$ has weight d and exponents $\alpha_1, ..., \alpha_n$. Then for every non-archimedean valuation v of K, the number of solutions $N_j(F,v)$ of the congruence*

$$F(x) \equiv 0 \ \ mod \ \ \pi^j \mathcal{O}_{K_v},$$

where $\mathcal{O}_{K_v}$ is the ring of integers of the completion K_v, satisfies

$$lím \ sup_{j\to\infty} N_j(F,v)^{1/j} \le \begin{cases} q^{n-|\alpha|/d} & if \ \mid \alpha \mid /d \le 1, \\ q^{n-1} & if \ \mid \alpha \mid /d > 1. \end{cases}$$

The following corollary follows directly from theorem 3.5. We use the notation established in the introduction.

Corollary 3.8. *Let $f(x,y) \in K[x,y]$ be an absolutely analytically irreducible polynomial, such that the origin of K^2 is an absolutely algebraically isolated singularity of the plane curve V_{f_d}. Then the Igusa local zeta function $Z(f,s)$ is a rational function of q^{-s} of form (3.15).*

Example 3.9. In this example we compute the local zeta function for a polynomial of type $f(x,y) = \alpha x^n + \beta y^m$, $\alpha, \beta \in \mathcal{O}_K$, where $n, m > 1$ are relatively prime. Suppose that the characteristic of K does not divide both n and m. Furthermore, without loss of generality, we may suppose that $\alpha \in \mathcal{O}_K^*$. In the case of charactersitic zero, the Poincaré series $P(t)$ associated to this type of polynomials were explicitly computed by Goldman (cf. [6], thm. 1).

Using the observation made after the proof of theorem 3.5, we have

$$Z(f,s) = \frac{1}{1-q^{-(n+m)-mns}} \int_{A^c} |f|_K^s \mid dxdy \mid, \tag{3.19}$$

where $A = \{(x,y) \in \mathcal{O}_K^2 \ \mid \ v(x) \ge m, \ \ v(y) \ge n\}$. The complement A^c of A is the disjoint union of the following three sets:

$$D_1 = \{(x,y) \in \mathcal{O}_K^2 \ \mid \ v(x) < m, \ \ v(y) \ge n\},$$

$$D_2 = \{(x,y) \in \mathcal{O}_K^2 \mid v(x) < m, \quad v(y) < n\}.$$

$$D_3 = \{(x,y) \in \mathcal{O}_K^2 \mid v(x) \geq m, \quad v(y) < n\}.$$

Thus from (3.19), we obtain

$$Z(f,s) = \frac{1}{1-q^{-(n+m)-mns}} \left\{ Z(f,D_1,s) + Z(f,D_2,s) + Z(f,D_3,s) \right\}. \tag{3.20}$$

Next, we compute the integrals $Z(f,D_1,s)$, $Z(f,D_2,s)$, $Z(f,D_3,s)$.

Computation of $Z(f,D_1,s)$**.** First, we observe that

$$\mid f(x,y) \mid = \mid \alpha x^n + \beta y^m \mid = \mid x^n \mid, \qquad x, y \in D_1.$$

Therefore

$$Z(f,D_1,s) = \int_{D_1} |f|_K^s \mid dxdy \mid = \sum_{k=0}^{m-1} \int_{\{(x,y)\in D_1 \ \mid \ v(x)=k,\, v(y)\geq n\}} |x|_K^{ns} \mid dxdy \mid.$$

Thus

$$Z(f,D_1,s) = (1-q^{-1})q^{-n} \sum_{k=0}^{m-1} q^{-kns-k} \tag{3.21}$$

Computation of $Z(f,D_2,s)$**.** We set $L(i,j) := jm - in + v(\beta)$. The set D_2 can be decomposed as the union of three disjoint subsets $D_{2,1}, D_{2,2}, D_{2,3}$, as follows:

$$D_{2,1} := \{(x,y) \in D_2 \mid L(v(x),v(y)) > 0\},$$

$$D_{2,2} := \{(x,y) \in D_2 \mid L(v(x),v(y)) < 0\},$$

$$D_{2,3} := \{(x,y) \in D_2 \mid L(v(x),v(y)) = 0\}.$$

Thus $Z(f,D_2,s) = Z(f,D_{2,1},s) + Z(f,D_{2,2},s) + Z(f,D_{2,3},s)$, where

$$Z(f,D_{2,1},s) = (1-q^{-1})^2 \sum_{i,j} q^{-i-j-nis}, \tag{3.22}$$

where i,j satisfy $L(i,j) > 0$ and $0 \leqq i < m$, $0 \leqq j < n$,

$$Z(f,D_{2,2},s) = (1-q^{-1})^2 \sum_{i,j} q^{-i-j-(v(\beta)+mj)s}, \tag{3.23}$$

where i,j satisfy $L(i,j) < 0$ and $0 \leqq i < m$, $0 \leqq j < n$, and

$$Z(f,D_{2,3},s) = \sum_{i,j} q^{-i-j-nis} \int_{\mathcal{O}_K^{\times\,2}} |\alpha x^n + \mu y^m|_K^s \mid dxdy \mid, \tag{3.24}$$

where $\beta = \pi^{v(\beta)}\mu$, $\mu \in \mathcal{O}_K^\times$, i,j satisfy $L(i,j) = 0$ and $0 \leqq i < m$, $0 \leqq j < n$. Using the stationary phase formula, we compute the integral in the right side of (3.24); thus

$$Z(f,D_{2,3},s) = \left(\nu(\overline{f}, \mathcal{O}_K^{\times 2}) + \frac{\sigma(\overline{f}, \mathcal{O}_K^{\times 2})(1-q^{-1})q^{-s}}{1-q^{-1-s}} \right) \sum_{i,j} q^{-i-j-nis}.$$

We denote by $[x]$ the integer part of a real number x. We set $v(\beta) = gn + r$, $0 \leq r < n$.

Computation of $Z(f, D_3, s)$. We set

$$D_{3,1} := \{(x,y) \in \mathcal{O}_K^2 \ \mid \ v(x) \geq m + [\frac{v(\beta)}{n}] + 1, \ \ v(y) < n\},$$

$$D_{3,2} := \{(x,y) \in \mathcal{O}_K^2 \ \mid \ m \leq v(x) \leq m + [\frac{v(\beta)}{n}], \ \ v(y) < n\}.$$

Then $D_3 = D_{3,1} \cup D_{3,2}$ (disjoint union), and $Z(f, D_3, s) = Z(f, D_{3,1}, s) + Z(f, D_{3,2}, s)$. To compute $Z(f, D_{3,1}, s)$, we observe that

$$|f(x,y)| = |\alpha x^n + \beta y^m| = |\beta y^m|, \qquad x, y \in D_{3,1}.$$

Thus

$$\begin{aligned} Z(f, D_{3,1}, s) &= \int_{D_{3,1}} |f(x,y)|_K^s \mid dxdy \mid \\ &= (1 - q^{-1}) q^{-(m + [\frac{v(\beta)}{n}] + 1)} \sum_{k=0}^{n-1} q^{-(v(\beta) + mk)s - k}. \end{aligned} \tag{3.25}$$

The set $D_{3,2}$ can be decomposed as the union of three disjoint subsets $D_{3,2,1}$, $D_{3,2,2}$, $D_{3,2,3}$, as follows:

$$D_{3,2,1} := \{(x,y) \in D_{3,2} \ \mid \ L(v(x), v(y)) > 0\},$$

$$D_{3,2,2} := \{(x,y) \in D_{3,2} \ \mid \ L(v(x), v(y)) < 0\},$$

$$D_{3,2,3} := \{(x,y) \in D_{3,2} \ \mid \ L(v(x), v(y)) = 0\}.$$

Thus $Z(f, D_{3,2}, s) = Z(f, D_{3,2,1}, s) + Z(f, D_{3,2,2}, s) + Z(f, D_{3,2,3}, s)$, and

$$Z(f, D_{3,2,1}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-nis}, \tag{3.26}$$

where i, j satisfy $L(i,j) > 0$ and $m \leqq i < m + [\frac{v(\beta)}{n}]$, $0 \leqq j < n$,

$$Z(f, D_{3,2,2}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-(v(\beta)+mj)s}, \tag{3.27}$$

where i, j satisfy $L(i,j) < 0$ and $m \leqq i < m + [\frac{v(\beta)}{n}]$, $0 \leqq j < n$, and

$$Z(f, D_{3,2,3}, s) = \left(\nu(\overline{f}, \mathcal{O}_K^{\times 2}) + \frac{\sigma(\overline{f}, \mathcal{O}_K^{\times 2})(1 - q^{-1}) q^{-s}}{1 - q^{-1-s}} \right) \sum_{i,j} q^{-i-j-nis},$$

where i, j satisfy $L(i,j) = 0$ and $m \leqq i < m + [\frac{v(\beta)}{n}]$, $0 \leqq j < n$.

Example 3.10. A polynomial of the form $f(x) = \sum_i \beta_i x_i^{n_i}$, $\beta_i \in \mathcal{O}_K$, is called a diagonal polynomial. We set $d := l.c.m.\{n_i\}$, and $\alpha_i := \frac{d}{n_i}$, $i = 1, .., n$. If the characteristic of K does not divide any n_i, then the diagonal polynomials are quasihomogeneous polynomials with exponents $\alpha_i := \frac{d}{n_i}$, $i = 1, .., n$, and weight d. Thus the local zeta function of a diagonal polynomial is a rational function of the form (3.13). Wang and others have studied the Poincaré series $P(t)$ associated to this class of polynomials (cf. [14], thm. 1).

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